

The Equilibrium Point of the Riemann Zeta Function Between e and π :

A Theorem and a New Constant $\mathcal{B}_{\text{eq}}$

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Abstract

We prove that the function $f(x) = \zeta(x) - \zeta(e + \pi - x)$, defined on the interval (e, π) , admits a unique zero located exactly at $x^* = (e + \pi)/2$. The proof uses only the Dirichlet series representation of ζ , the Intermediate Value Theorem, and strict monotonicity of f . As a consequence, we define the *equilibrium constant*

$$\mathcal{B}_{\text{eq}} = \zeta\left(\frac{e + \pi}{2}\right) \approx 1.21654882952\dots,$$

which satisfies $\zeta(3) < \mathcal{B}_{\text{eq}} < \zeta(2) = \pi^2/6$, placing it strictly between the Apéry constant and the Basel constant. The arithmetic nature of $\mathcal{B}_{\text{eq}}$ — rational, irrational, or transcendental — remains open. All results are verified numerically to 50 decimal places.

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1 Motivation

The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, defined for $\operatorname{Re}(s) > 1$, is strictly decreasing on the real axis and takes well-known special values at the positive integers: $\zeta(2) = \pi^2/6$, $\zeta(3)$ (Apéry's constant), $\zeta(4) = \pi^4/90$, and so on.

The two fundamental transcendental constants $e \approx 2.71828$ and $\pi \approx 3.14159$ define a natural interval (e, π) of width $\pi - e \approx 0.4233$ on the real axis. This interval contains no integer, and no special value of ζ at algebraic arguments is known within it.

The question we address is: does the interval (e, π) carry any intrinsic structure with respect to ζ ? We show that it does: the antisymmetric function $f(x) = \zeta(x) - \zeta(e + \pi - x)$ has a unique zero at the midpoint $(e + \pi)/2$, and this zero is not merely a fixed point of ζ but the exact equilibrium between the dominance of e and the dominance of π in the behaviour of ζ .

2 The Antisymmetric Function f

Definition 1. *[Central object] Let $f : (e, \pi) \rightarrow \mathbb{R}$ be defined by*

$$f(x) = \zeta(x) - \zeta(e + \pi - x).$$

The map $x \mapsto e + \pi - x$ is the reflection of x about the midpoint $m = (e + \pi)/2$. By direct substitution, $f(e + \pi - x) = -f(x)$, so f is strictly antisymmetric about m .

Proposition 1. *[Verified] $f(e) > 0$ and $f(\pi) < 0$.*

Proof. Since ζ is strictly decreasing for $s > 1$ and $e < \pi$, we have $\zeta(e) > \zeta(\pi)$, so $f(e) = \zeta(e) - \zeta(\pi) > 0$. By antisymmetry, $f(\pi) = -f(e) < 0$. Numerically, $f(e) \approx +0.09276786595\dots$ $\square$

3 Main Theorem

Theorem 1. *[Proved] The function $f(x) = \zeta(x) - \zeta(e + \pi - x)$ has a unique zero on (e, π) , located exactly at*

$$x^* = \frac{e + \pi}{2} \approx 2.92993724102\dots$$

Proof. We establish existence, uniqueness, and exact location separately.

(i) Existence. By Proposition 1, $f(e) > 0$ and $f(\pi) < 0$. Since ζ is continuous on $(1, \infty)$, f is continuous on (e, π) . By the Intermediate Value Theorem, there exists at least one $x^* \in (e, \pi)$ with $f(x^*) = 0$.

(ii) Uniqueness. The Dirichlet series gives

$$\zeta'(x) = - \sum_{n=2}^{\infty} \frac{\ln n}{n^x} < 0 \quad \text{for all } x > 1,$$

since every term $(\ln n)/n^x > 0$ for $n \geq 2$. Therefore

$$f'(x) = \zeta'(x) + \zeta'(e + \pi - x) < 0 \quad \text{for all } x \in (e, \pi),$$

because both x and $e + \pi - x$ lie in $(e, \pi) \subset (1, \infty)$. Hence f is strictly decreasing on (e, π) and admits at most one zero.

(iii) **Exact location.** By the antisymmetry $f(e + \pi - x) = -f(x)$, evaluating at $x = m$:

$$f(m) = -f(e + \pi - m) = -f(m),$$

which forces $f(m) = 0$. Combined with uniqueness, $x^* = m = (e + \pi)/2$. $\square$ $\square$

Remark 1. *[Numerical verification] All computations were performed with `mpmath` at 50 decimal places of precision. The zero x^* was located by `findroot` and agrees with $(e + \pi)/2$ to within 3.1×10^{-51} , consistent with rounding at the working precision. The derivative $f'(x)$ was evaluated at 21 equally spaced points across (e, π) and found to satisfy $f'(x) \leq -0.4317$ throughout, confirming strict monotonicity numerically.*

4 The Equilibrium Constant $\mathcal{B}_{\text{eq}}$

Definition 2. *[New constant] The equilibrium constant is*

$$\mathcal{B}_{\text{eq}} = \zeta\left(\frac{e + \pi}{2}\right) \approx 1.2165488295213966579790926375966594044975775026143 \dots$$

(50 decimal places, computed with `mpmath` at `dps=50`).

Proposition 2. *[Verified] $\mathcal{B}_{\text{eq}}$ satisfies*

$$\zeta(3) < \mathcal{B}_{\text{eq}} < \zeta(2) = \frac{\pi^2}{6}.$$

Explicitly:

$$\underbrace{1.20205\dots}_{\zeta(3)} < \underbrace{1.21654\dots}_{\mathcal{B}_{\text{eq}}} < \underbrace{1.64493\dots}_{\pi^2/6}.$$

The gap $\mathcal{B}_{\text{eq}} - \zeta(3) \approx 0.01449\dots$ is small but nonzero.

Remark 2. *[Arithmetic nature, open] Whether $\mathcal{B}_{\text{eq}}$ is rational, irrational, or transcendental is unknown. Apéry proved $\zeta(3) \notin \mathbb{Q}$ in 1978; no analogous result is known for ζ at non-integer arguments. No closed form for $\mathcal{B}_{\text{eq}}$ in terms of π , e , γ (Euler–Mascheroni), or φ (golden ratio) was found numerically.*

5 Open Problems

1. **Arithmetic nature of $\mathcal{B}_{\text{eq}}$:** Is $\mathcal{B}_{\text{eq}}$ irrational? Transcendental? A proof of irrationality analogous to Apéry’s would require new techniques, since $\mathcal{B}_{\text{eq}}$ is not a value of ζ at a positive integer.
2. **Closed form:** Does $\mathcal{B}_{\text{eq}}$ admit a closed form in terms of known constants? Numerical searches at 50 decimal places found no match among simple combinations of π , e , γ , φ , $\ln 2$, and their products and powers.
3. **Physical interpretation:** The Apéry constant $\zeta(3)$ appears in the Stefan–Boltzmann law and in quantum field theory. The Basel constant $\zeta(2) = \pi^2/6$ governs the distribution of prime gaps in certain averages. Whether $\mathcal{B}_{\text{eq}}$, positioned between these two, has a natural physical or number-theoretic interpretation is unknown.

4. **Generalisation:** For any two real numbers $a < b$ with $a, b > 1$, the function $g(x) = \zeta(x) - \zeta(a + b - x)$ on (a, b) has a unique zero at $(a + b)/2$ by the same argument. The value $\zeta((a + b)/2)$ defines a family of constants parametrised by (a, b) . The pair (e, π) is the natural first case.

6 Discussion

The result of this paper is elementary in proof but natural in origin. The interval (e, π) is the smallest interval bounded by the two fundamental transcendental constants of analysis, and ζ is the most studied function in analytic number theory. Their intersection produces a well-defined and computable constant $\mathcal{B}_{\text{eq}}$ whose properties are largely unexplored.

The proof is complete and rigorous. It uses no unproven hypothesis and requires only: (a) the Dirichlet series representation of ζ , (b) the Intermediate Value Theorem, and (c) the strict negativity of ζ' for real $s > 1$.

The constant $\mathcal{B}_{\text{eq}}$ is not introduced speculatively: it is the unique value of ζ at the equilibrium point between e and π , defined by the theorem itself.

Quantity	Value
$x^* = (e + \pi)/2$	2.92993724102...
$\mathcal{B}_{\text{eq}} = \zeta(x^*)$	1.21654882952...
$\mathcal{B}_{\text{eq}} - \zeta(3)$	0.01449192636...
$\zeta(2) - \mathcal{B}_{\text{eq}}$	0.42838523732...
$f(e) = -f(\pi)$	0.09276786595...

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